Numerical multiscale methods for a reaction-dominated model

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A B S T R A C T

A Galerkin enriched finite element method (GEM) is proposed for the singularly perturbed reaction–diffusion equation. This new method is an improvement on the Petrov–Galerkin enriched method (PGEM), where now the standard piecewise (bi)linear test space incorporates fine scales. This appears as the fundamental ingredient for suppressing oscillations in the numerical solutions. Also, new parameter-free stabilized finite element methods derived from both the GEM and the PGEM are driven by local generalized eigenvalue problems. In the process, jump stabilizing terms belonging to the class of CIP methods emerge as a result of the enriching procedure. Interestingly, numerical results indicate that jump-based stabilizations are unnecessary and sometimes undesirable when treating reaction-dominated problems. Finally, we establish relationships with more standard enriched and stabilized methods and show that the proposed methods outperform them numerically.

1. Introduction

Aris ing from realistic modeling in diverse fields such as biology (ecosystem or neuroscience models), chemistry (reactive processes), or in mechanical engineering (transport in porous media, turbulence model or wildfire problems, just to cite a few) is the reaction-dominated reaction–diffusion equation. Solutions to such problems exhibit boundary layers which must be handled carefully by numerical schemes. Without such care, approximation of the gradient of the true solution in the natural norm is adversely affected, leading to non-physical oscillations in the numerical solution. For example, solving this singularly perturbed model using the Galerkin method based on local polynomial interpolation requires specially refined meshes (Shishkin meshes [1], for instance) in regions containing boundary layers in order to avoid non-physical oscillations. However, such an approach becomes increasingly prohibitive for real-life problems defined on complex geometries and demands prior knowledge of the position of sharp layers. In this work, we pursue the idea of constructing robust finite element methods for solving reaction-dominated problems on coarse (un)structured meshes. Among the alternatives is the stabilized finite element method [2]. When applied to the reactive–diffusive model, numerical solutions can fulfill the maximum principle property on coarse meshes. The general idea is to select appropriate weak stabilizing terms to be included in the Galerkin formulation. Several possibilities that modify the original formulation exist, among which are the addition of weak terms depending on the gradient of the residual of the Lagrange equation [3–5] or the subtraction of weak terms dependent on the residual of the strong model [6,7] (see [8] for a comparison of the performance between several stabilized methods). In an attempt to build operator-independent stabilized methods, edge-based jump terms have been used to penalize inter-element fluxes [9–11]. These stabilization alternatives are balanced by parameters which must be carefully tuned in advance, as their values have a great impact on the accuracy of solutions.

The relationship of these stabilized methods with the Variational Multiscale approach (VMS) [12] has been investigated to better understand the mechanism governing the design of the stabilization parameters. The Residual-Free Bubble (RFB) method [13,14] and the Petrov–Galerkin enriched (PGEM) methods [15,16] are examples of such an approach (see [17] for an overview), wherein the standard polynomial space is enriched with functions which capture subgrid behavior. Such enrichment functions are computed from local boundary value problems arising from a static condensation procedure. Thereby, the unsolved fine scales are represented in terms of coarse ones through element-wise residuals. Furthermore, edge residuals play an important role in the construction of enriching basis functions for the reaction–diffusion model [15]. This is, actually, the PGEM’s distinguishing feature (see [18] for an overview) and allows the method to produce solutions which are accurate in natural norms. Other approaches rely on nonstandard polynomial basis functions, such as the PUM (Partition of Unity Method). See [19] for a recent overview.

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This work presents a threefold contribution. First, a new Galerkin enriched finite element method (GEM) is proposed by modifying the PGEM so that the test space is now defined so as to include fine scales. Such a feature was, indeed, restricted to the trial space in [15]. Readers are invited to consider [20,21], where such a strategy was introduced by observing the presence of fine scales in the test space in 1D leads to nodal exactness (analogous to what occurs in the oscillatory elliptic case [22]). Numerical results show that the new method outperforms the original PGEM when the reaction (balanced by the element diameter) and diffusion coefficients are derived inside an enriching space strategy (see [25] and [26]). The upshot of the approach is that no stabilization parameters are fixed through the action of the multiscale basis functions. This is a well-known strategy in the context of bubble functions. This is equivalent to that of CIP methods. Furthermore, the subspace $H^2_0(D) \subset H^1(D)$ consists of functions with support in $D$, $H^{-1}(D)$ stands for the dual space of $H^1_0(D)$, and the mean value of function $v \in L^2(D)$ is denoted by $\Pi_D(v) := \frac{1}{|D|} \int_D v$. However, this relationship between some well-established numerical strategies and the construction of enriching basis functions is considered. Namely, we draw comparisons with the edge-based Continuous Interior Penalty method (CIP) [10] and the multiscale finite element method (MsFEM) [24]. As for the former, the approach leads to a perturbation of the GEM with boundary terms that explicitly control the jump of the flux in a form equivalent to that of CIP methods.

To the best of the authors’ knowledge, this is the first time edge-based finite element methods for the reactive–diffusive equation are derived inside an enriching strategy (see [25] and [26]) for a related idea applied to the Darcy and Stokes models, respectively. The upshot of the approach is that no stabilization parameter needs to be fixed. Also, we show that the most suitable MsFEM version in [24] may be recovered from the present enhancing idea with a modified right hand side. Numerical tests attest that this difference is fundamental to greatly reduce resonance errors.

Finally, stabilized finite element methods derived from the GEM and the PGEM [15] are introduced. These new methods are of the unusual finite element type [6], in which the elemental stabilization parameters are fixed through the action of the multiscale basis functions. This is a well-known strategy in the context of bubble function enrichment for advective–diffusive problems [27,28]. Following the methodology [29] of producing stabilization parameters through enrichment functions, this idea is extended to reaction-dominated problems using local generalized eigenvalue problems. The GEM and its stabilized method counterpart are numerically compared with more classical stabilized methods in [24], and is explained by the lack of convergence with respect to the gradient of the solutions for intermediate values of diffusive coefficient.

Second, the relationship between some well-established numerical tests are found in Section 7, followed by conclusions in Section 8.

1.1. Notation and preliminaries

This section introduces definitions and notation used throughout. Let $D$ be a bounded set in $\mathbb{R}^d$, $d \in \{1,2\}$ and suppose $L^2(D)$ is the usual space of square integrable functions over $D$. This space is equipped with an inner-product $\langle \cdot, \cdot \rangle$. The space $H^2(D)$ is the subspace of $L^2(D)$ whose functions have square integrable derivatives up to order 2. The $H^2(D)$ semi-norm is defined by $| \cdot |_{2,s}$.
into the domain as the Galerkin method seeks the weakly enforced global balance required by its definition. This Gibbs-like phenomenon demonstrates that the Galerkin method is unable to adequately approximate the true solution in the natural space $H^1(\Omega)$. The incorporation of fine scales through the action of operator $\rho(\cdot)$ inserts the right dose of numerical diffusion for problems with vanishing $\varepsilon$, resulting in highly improved numerical solutions (see Fig. 2, which shows an oscillation-free PGEM solution). Such a feature was mathematically analyzed in [16], wherein a uniform convergence result with respect to $\varepsilon$ was proved under the condition that the characteristic length of elements intersecting $\partial\Omega$ is of order $\varepsilon^{1/2}$. In spite of this marked improvement, not fulfilling such a condition permits some non-physical oscillations to persist for certain values of $\varepsilon$ (see Fig. 3). This phenomenon, known as resonance, results from an improper treatment of the $H^1(\Omega)$ semi-norm which occurs in this case when $h$ and $\varepsilon^{1/2}$ are of the same order of magnitude (see [24] for details in multiscale coefficient elliptic models). First pointed out numerically in [23] for reaction–diffusion problems, this drawback is proved to be localized near $\partial\Omega$ and may be overcome by refining the mesh near the boundary [16]. In the next section, a mesh independent alternative to this approach is presented, noting that the PGEM incorporates fine scales only in the trial space.

3. The Petrov–Galerkin enriched framework

The aim of this section is to present the GEM, which modifies the PGEM in order to maintain its strong $H^1(\Omega)$ approximation properties while reducing the effect of the resonance error. The way in which enrichment functions are selected in the PGEM is intimately related to its success in capturing boundary layers. As such, [15] is followed closely in the sense that the polynomial space $V_h(\Omega)$ is enhanced with special functions which maintain the degrees of freedom of $V_h(\Omega)$. For the purpose of exposition only, we continue to assume $\sigma f \in R$ in $\Omega$.

Denote the enriched trial and test spaces by $V_h(\Omega)$ and $W_h(\Omega)$, respectively. To define them, we follow a constructive process which begins by looking for $u_h \in V_h(\Omega)$ in the form

$$u_h = u_1 + u_e,$$

where $u_e$ resolves fine scales through local problems based on residuals of the method. The effect of $u_e$ can be made explicit. First, suppose the edge-based operator $L_{\text{edge}}$ is defined by

$$L_{\text{edge}} g := \sigma g - u_{\partial K} g \quad \text{on} \quad F \subset \partial K \in \mathcal{E}_h,$$  \hspace{1cm} (6)

where $\sigma$ is a variable that parametrizes an edge $F$ by arc-length, and $c$ a positive constant which will be fixed later on (see Remark on Page 10). Next, consider the operator $\rho : V_h(\Omega) \rightarrow H^1(\Omega)$ with the property that for each $K \in T_h$ and $u_1 \in \mathcal{S}_1(K)$, the restriction of $\rho(\cdot)$ to $K$ is given by $\rho(u_1)_{|K} := w_h$, where

$$L_{\text{edge}} w_h = 0 \quad \text{in} \quad K, \quad w_h = g \quad \text{on} \quad \partial K$$

and $g \in H^1(\partial K)$ satisfies

$$L_{\text{edge}} g = 0 \quad \text{on} \quad F \subset \partial K, \quad g = w_1 \quad \text{at the nodes}$$

for $F \in \mathcal{E}_e$, and $g = w_1$ on $\partial \Omega$.

The idea behind the choice of boundary condition in (7) is that the solution on the edge should present the same qualitative behavior as the one inside the element. It follows that a 1D version of the reaction–diffusion operator on the edges would be a fair guideline to build it. This idea was first introduced in [24] for the Poisson problem with oscillatory coefficients, and is shown to be a better choice than the homogenous Dirichlet condition used in the RFB method in [15].

It is clear from this definition that the image space $\rho(V_h(\Omega))$ is driven by the degrees of freedom associated with $V_h(\Omega)$, and is therefore finite dimensional. The function $u_e$ is given in terms of problem (7) (see [15] for details), namely,

$$u_e_{|K} = \frac{1}{\sigma}(I - \rho)(f - \sigma u_1),$$

where $I$ is the identity operator.

Upon updating the linear function $u_1$ with $u_e$ in the Galerkin method, the coarse scale approximation $u_1$ is obtained from the following PGEM: Find $u_1 \in V_1(\Omega)$ such that:

$$a(\rho(u_1), v_1) = (f, v_1)_{|\Omega} - \frac{1}{\sigma}a(f - \rho(f), v_1) \quad \text{for all} \quad v_1 \in V_1(\Omega).$$

(9)

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Denote the enriched trial and test spaces by $V_h(\Omega)$ and $W_h(\Omega)$, respectively. To define them, we follow a constructive process which begins by looking for $u_h \in V_h(\Omega)$ in the form

$$u_h = u_1 + u_e,$$

where $u_1 \in V_1(\Omega)$. As no indication is available at this point, we search for $u_e$ in the whole space $H^1_0(\Omega)$ for now.

Next, we note that $\rho(V_h(\Omega))$ set as the test space induces nodal-exactness in the one-dimensional case (seen as a straight...
adaptation of Remark 4.2 in [22] to the present case). Hence, we select \( \rho(\psi_1) \) (instead of \( V_1(\Omega) \)) to be enhanced with functions with local support in each \( K \in T_h \); thereby, a static condensation procedure can be employed. In other words, the test space \( W_h(\Omega) \) incorporates fine scales and is defined by

\[
W_h(\Omega) := \rho(V_1(\Omega)) \oplus H^0_\sigma(T_h).
\]

(11)

Accordingly, an element \( v_h \in W_h(\Omega) \) may be decomposed as

\[
v_h = \rho(u_1) + v_h,
\]

where \( u_1 \in V_1(\Omega) \) and \( v_h \in H^0_\sigma(T_h) \).

The associated Petrov–Galerkin formulation then reads: Find \( v_h \in V_h(\Omega) \) such that

\[
a(u_h, v_h) = (f, v_h)_{\Omega} \quad \text{for all} \quad v_h \in W_h(\Omega).
\]

(12)

Using the decomposition (11) in the definition of space \( W_h(\Omega) \), \( u_h \in \rho(V_1(\Omega)) \) clearly satisfies the equivalent system:

\[
a(u_h, v_1) = (f, v_1)_{\Omega} \quad \text{for all} \quad v_1 \in V_1(\Omega),
\]

(13)

\[
a(u_h, v_E^g) = (f, v_E^g)_{\Omega} \quad \text{for all} \quad v_E^g \in H^0_\sigma(K).
\]

(14)

Integrating (14) by parts, together with the condition \( v_E^g|_{\partial K} = 0 \) and \( \Delta u_1 = 0 \), shows the function \( u_h \) satisfies

\[
L u_h = f - \sigma u_1 \quad \text{in each} \quad K \in T_h.
\]

(15)

As the local operator inside elements is fixed, we are now left to prescribe boundary conditions for it. Uniqueness is assured for (15) by prescribing a non-homogenous Dirichlet boundary condition on internal edges given as follows:

\[
u_k = g \quad \text{on} \quad F \subset \partial K \setminus \partial \Omega
\]

(16)

and \( u_0 = 0 \) on \( \partial \Omega \), where \( g \in H^1(\partial K) \) satisfies

\[
L_{\partial K} g = \frac{\sigma}{\alpha} (f - \sigma u_1) \quad \text{on} \quad F, \quad g = 0 \quad \text{at the nodes}.
\]

(17)

Remark. At this point, we can revisit \( V_h(\Omega) \) and refine its definition as follows

\[
V_h(\Omega) := V_1(\Omega)
\]

\[
\oplus \left\{ v_p \in H^0_\sigma(\Omega) : v_p|_{\partial K} = \frac{1}{\alpha} (1 - \rho)(f - \sigma v_1), \quad v_1 \in V_1(\Omega) \right\}
\]

which is a finite dimensional space since \( f \in \mathbb{R} \) is fixed and \( V_1(\Omega) \) is finite dimensional. \( \square \)

From Eqs. (15) and (16), it holds that \( u_1 \) is computed with respect to \( f \) and the degrees of freedom of its coarse-scale counterpart \( u_1 \). Following this observation, it is convenient to split \( u_1 = u_1' + u_2 \), and \( g = g_1 + g_2 \), where each contribution satisfies,

\[
L u_1' = -\sigma u_1 \quad \text{in} \quad K, \quad u_1' = g_1 \quad \text{on} \quad \partial K,
\]

(18)

\[
L_{\partial K} g_1 = -\sigma u_1 \quad \text{on} \quad F \subset \partial K, \quad g_1 = 0 \quad \text{at the nodes}
\]

(19)

and

\[
L u_2' = f \quad \text{in} \quad K, \quad u_2' = g_2 \quad \text{on} \quad \partial K,
\]

(20)

\[
L_{\partial K} g_2 = \sigma f \quad \text{on} \quad F \subset \partial K \quad g_2 = 0 \quad \text{at the nodes}.
\]

(21)

Now, using \( f \in \mathbb{R} \), a closed form for the solutions of (18)–(21) is available and given in terms of the operator \( \rho(\cdot) \) defined in (7) and (8). In fact, it holds

\[
u_1' = \frac{1}{\sigma}(f - \rho(f)) \quad \text{and} \quad u_2' = \rho(u_1) - u_1.
\]

(22)

Using the characterization above, we arrive at

\[
a(u_h, \rho(v_1)) = a(u_1 + u_1', \rho(v_1)) + a(u_2', \rho(v_1))
\]

\[
= a(\rho(u_1), \rho(v_1)) + a(u_2', \rho(v_1)).
\]

(23)

Gathering this together with the observation

\[
a(u_1', \rho(v_1)) = (f, \rho(v_1))_\Omega - \frac{1}{\alpha} a(\rho(f), \rho(v_1)),
\]

the GEM follows from (13) and reads: Find \( u_1 \in V_1(\Omega) \) such that

\[
a(\rho(u_1), \rho(v_1)) = \frac{1}{\alpha} a(\rho(f), \rho(v_1)) \quad \text{for all} \quad v_1 \in V_1(\Omega).
\]

(24)

Relaxing the assumptions on \( f \) and \( \sigma \) makes the characterization for \( u_h \) in (22) no longer valid. In such a case, the GEM is obtained replacing (23) in (13), which gives

\[
a(\rho(u_1), \rho(v_1)) = (f, \rho(v_1))_\Omega - a(u_2', \rho(v_1)) \quad \text{for all} \quad v_1 \in V_1(\Omega).
\]

(25)

where \( u_2' \) satisfies (20) and must be approached through a two-level method. Observe that (24) (or (25)) reflects the effect of the enhancing methodology (12), leading to a redesign of the basis functions.

Remarks.

- The difference between the PGEM (9) and the GEM (24) is the space of test functions. Whereas the test functions are (bi)linear for the former, multiscale aspects are incorporated in the latter. The numerical tests in Section 7 indicate this is key to minimizing the resonance error.
- The efficiency of method (24) arises from the ability to solve local problems (15) and (16) analytically (as opposed to a two-level approach [31]). Indeed, thanks to the linearity of operators \( \mathcal{L} \) and \( \mathcal{L}_{\partial K} \), what is needed is \( \rho(\psi_1) \), where \( \psi_1 \) represents the (bi)linear basis function associated with node 1. Following [15,16], a closed formula is made available by selecting the constant \( c \) appropriately in (16). On triangular meshes and rectilinear rectangular meshes, respectively, the following results hold:

\[
\rho(\psi_1) := \frac{\sinh(x_1 \psi_1)}{\sinh(x_1)} \quad \text{and}
\]

\[
\rho(\psi_1) := \frac{\sinh(x_2 \psi_1)}{\sinh(x_2)} \frac{\sinh(x_1 \psi_1)}{\sinh(x_1)},
\]

(26)

where \( \psi_1(x, y) \) defines the bilinear basis functions and \( x_u := \sqrt{x_1^2 + x_2^2}, \quad x_1 := \sqrt{x_1^2}, \quad x_2 := \sqrt{x_2^2} \), and \( h_u \) and \( h_y \) are the edge lengths of the rectangle \( K \). We recall that \( \sigma := c \sigma \), and then, \( c \)
equals $\frac{1}{2}$ for the rectilinear rectangular element, for example (see [15] for the exact value of $c$ in the triangle element case).

- Once $u_1$ is computed from (24), the enriching function $u_4$ is given analytically using the basis functions $\rho(\psi_4)$. It is then easy to calculate $u_{ih}$, which satisfies the point-wise conservation property $L u_{ih} = f$ in each $K$.

For rough $f \in H^1(\Omega)$, the enriching function $u_4$ must be approximated by a two level method. However, this extra computation may be avoided by replacing $u_4$ by $u_{4h}$ defined as the solution of (20) and (21), where the right-hand side is given by $H_h(f)$. Then, $u_{ih}$ rather than $u_4$ is used in the GEM to obtain the solution $u_1^h$.

Using (15) and (16), the result is $\rho(\bar{u}_1) + \bar{u}_h$, which satisfies the local conservation property

$$
\int_K L(\rho(\bar{u}_1) + \bar{u}_h) = \int_K f \quad \text{in each} \quad K \in T_h,
$$

without impacting accuracy. It is also worth pointing out that, unlike discontinuous Galerkin methods [32], the stated local conservation properties are reached with the relatively low cost of a continuous method.

- The boundary condition (16) implies $u_1$ is a conforming approximation of $u$, i.e.,

$$
\|u_1\|_0 = 0 \quad \text{for all} \quad F \in \mathcal{E}_h \quad \text{and} \quad u_1|_{F = \partial F} = u_1,
$$

once $c$ in $\sigma$ is uniquely prescribed on each $F \in \mathcal{E}_h$. Thereby, the basis functions (26) match on edges when quadrilateral and equivalent triangular meshes are used, and we assume this holds from now on. Furthermore, integrating Eq. (13) by parts, and using characterization (15) for $u_1$, it holds

$$
\sum_{K \in T_h} \int_K (\sigma \nabla u_1 \cdot n_1, \rho(v_1)) = \int_{\partial \Omega} L u_{ih} - f, \rho(v_1) |_{\partial \Omega} + \sum_{K \in T_h} (\sigma \nabla u_1 \cdot n_1, \rho(v_1))_{\partial K} = a(u_1, \rho(v_1)) - \int_{\partial \Omega} f, \rho(v_1) |_{\partial \Omega} = 0.
$$

Thus, the continuity of the flux is weakly imposed. This is the type of continuity enforcement which drives the construction of discontinuous Galerkin methods (see [33] for an interesting viewpoint).

- Problems with varying $\sigma$ and $\varepsilon$ are handled analytically by solving problems (18)–(21) where $H_h(\sigma)$ and $H_h(\varepsilon)$ replace the original reactive and diffusive coefficients. No loss of accuracy is expected as the consistency error relies on the approximation property of $H_h(\cdot)$ and, thus, does not affect convergence properties of the method. As for the 3D extension, we may adapt the boundary condition constant $c$ and still propose the use of the basis functions $\rho(\psi_4)$ given above. We point out that in both cases the modification is concerned with the redesign of $\mathbf{s}_K$ only. \hfill $\square$

4. Relationship with the MsFEM

Recall the observation that the space $\rho(V_1(\Omega))$ as defined by (7) is finite dimensional, consisting of basis functions taking degrees of freedom of $V_1(\Omega)$. Observing this basis contains subgrid information, [24,22] introduced the Multiscale Finite Element Method (MsFEM): Find $u_1 \in V_1(\Omega)$ such that

$$
a(\rho(u_1), \rho(v_1)) = (f, \rho(v_1)) \quad \text{for all} \quad v_1 \in V_1(\Omega)
$$

(27)

for the case that $\varepsilon$ is a function with multiple scales and $\sigma = 0$. It is interesting to study how a method of this form performs when extended to a highly reactive reaction–diffusion equation and how this performance compares to the GEM [24].

In fact, the GEM is closely related to this method in form. We recall from (25) that the GEM reads: Find $u_1 \in V_1(\Omega)$ such that

$$
a(\rho(u_1), \rho(v_1)) = (f, \rho(v_1)) - a(u^e_h, \rho(v_1)) \quad \text{for all} \quad v_1 \in V_1(\Omega).
$$

(28)

To better understand the effect of the term on the right-hand side, decompose (20) and (21) as $u_4 = u_4^e + u_4^e$, where

$$
\nabla u_4^e = f \quad \text{in} \quad K, \quad u_4^e = 0 \quad \text{on} \quad \partial K
$$

and

$$
\nabla u_4^e = 0 \quad \text{in} \quad K, \quad u_4^e = g_f \quad \text{on} \quad \partial K, \quad g_f = 0 \quad \text{at the nodes}.
$$

(29)

(30)

By definition (7) of $\rho(v_1)$ and the fact that $u_4 \in H_0^1(T_h)$, we obtain

$$
a(u_4^e, \rho(v_1)) = 0,
$$

so that (28) becomes

$$
a(\rho(u^e_1), \rho(v_1)) = (f, \rho(v_1)) - a(u_4^e, \rho(v_1)) \quad \text{for all} \quad v_1 \in V_1(\Omega).
$$

(31)

This clearly shows that the portion of the element–based residual associated with the source term has no impact on the GEM. As it stands, the nonzero boundary condition in (29) implies an extra contribution to the GEM over that which is present in the MsFEM extended to the reaction–diffusion equation. In fact, integrating by parts, applying Eq. (29), and using the continuity of $\rho(v_1)$ implies

$$
a(\rho(u^e_1), \rho(v_1)) = \sum_{F \in \mathcal{E}_h} (\bar{u}_4^e \nabla u^e_1 \cdot n_1, \rho(v_1))_{\partial K}.
$$

Therefore, the ultimate contribution of $u_4^e$ is the penalization of jumps of the gradient on internal edges. This result is consistent with the remark in Section 3 regarding the observation that the GEM may be reduced to a method which imposes the weak continuity of the flux on internal edges. Numerical tests in Section 7 explore the effect of this edge-based contribution by drawing comparisons between the GEM and the MsFEM extended to the reaction–diffusion equation.

Remark. In [24,22] another version of the MsFEM was introduced where the functions are assumed to vary linearly on internal edges. This has been compared to the RFB method [34], where the difference arises in the RFB’s calculation of local enrichments related to the source term. As this term does not appear explicitly in the coarse problem, the RFB and MsFEM are equivalent in this case. However, numerical results in [15] have shown that the RFB method lacks precision when applied to reaction-dominated problems. \hfill $\square$

5. Recovering a CIP method

CIP methods have been used to stabilize singularly perturbed models, with an emphasis placed on highly advective problems. Although it has been claimed that they may be made to work in the reaction-dominated regime as well, to the best of the authors’ knowledge no CIP method has been numerically verified for this case. Bearing this in mind, the enriched approach presented in the last section may be used to derive new methods which, in particular cases, have a CIP-like form.

To this end, and mimicking the enriching procedure used to derive the GEM (24), consider the modified boundary condition for local problem (15) which penalizes the jump of the flux across internal edges:
and \( u_e = g \) on \( F \in \mathcal{E}_h \)

(32)

and \( u_e = 0 \) on \( \partial \Omega \), where \( g \) satisfies

\[
\mathcal{L}_{\partial \Omega} g = \frac{\sigma}{\rho} (f - \sigma u_e) + \frac{e}{h_F} \| \nabla u_e \cdot n \| \quad \text{on} \quad F \subset \partial \Omega, \quad g = 0 \quad \text{at the nodes.}
\]

(33)

This choice agrees with the \textit{a posteriori} error estimators proposed for elliptic problems in [35], wherein the jump of the flux is interpreted as the residual across edges. Consequently, the function \( u_e \) now incorporates an enriching contribution, denoted \( u_e^r \), which accounts for the jump of the flux and solves

\[
\mathcal{L}_{\partial \Omega} u_e^r = 0 \quad \text{in} \quad K, \quad u_e^r = g_j \quad \text{on} \quad \partial K, \quad g_j = 0 \quad \text{at the nodes.}
\]

(34) and (35)

Computed from (18)–(21) and (34) and (35), the redefined \( u_e \) still fulfills Eq. (14). The boundary condition \( g_j \) may be rewritten as \( g_j = b_j \frac{e}{h_F} \| \nabla u_e \cdot n \| \), where an analytic expression for \( b_j \) is easily obtained by solving

\[
\mathcal{L}_{\partial \Omega} b_j = 1 \quad \text{on} \quad F \subset \partial \Omega, \quad b_j = 0 \quad \text{at nodes.}
\]

(36)

From (7), \( \mathcal{L}_e \rho (v_1) = 0 \) in each \( K \), which indicates the contribution of \( u_e^r \) in the method may be rewritten as

\[
a(u_e^r, \rho (v_1)) = \sum_{K \in \mathcal{T}_h} (u_e^r, L \rho (v_1))_K + \sum_{K \in \mathcal{T}_h} (u_e^r, e \nabla \rho (v_1) \cdot n)_K
\]

\[
= \sum_{K \in \mathcal{T}_h} (g_j, e \nabla \rho (v_1) \cdot n)_F
\]

\[
= \sum_{K \in \mathcal{T}_h} \frac{e}{h_F} (b_j \| \nabla u_e \cdot n \|, e \| \nabla \rho (v_1) \cdot n \|)_F.
\]

(37)

The characterization above yields an alternative version of the GEM with an explicit control on the flux across internal edges: \textit{Find} \( u_1 \in V_1(\Omega) \) such that

\[
B(u_1, v_1) = F(v_1) \quad \text{for all} \quad v_1 \in V_1(\Omega),
\]

(38)

with

\[
B(u_1, v_1) := a(\rho (u_1), \rho (v_1)) + \sum_{K \in \mathcal{T}_h} \frac{e}{h_F} (b_F \| \nabla u_1 \cdot n \|, e \| \nabla \rho (v_1) \cdot n \|)_F,
\]

\[
F(v_1) := \frac{1}{\sigma} a(\rho (f), \rho (v_1)),
\]

where \( b_F \) solves (36).

\textbf{Remark.} Similarly, the PGEM of [15] may be updated with the jump-dependent enrichment (34) and (35), which leads to an alternative PGEM. Following (37), with \( v_1 \) replacing test functions \( \rho (v_1) \), and neglecting the term \( \sum_{K \in \mathcal{T}_h} (u_e^r, \sigma v_1)_K \), this alternative PGEM with flux control on the edges reads: \textit{Find} \( u_1 \in V_1(\Omega) \) such that

\[
B_{\text{PGEM}}(u_1, v_1) = F_{\text{PGEM}}(v_1) \quad \text{for all} \quad v_1 \in V_1(\Omega),
\]

(39)

with

\[
B_{\text{PGEM}}(u_1, v_1) := a(\rho (u_1), \rho (v_1)) + \sum_{F \in \mathcal{F}_h} \tau_F \| [\nabla v_1 \cdot n], e [\nabla \rho (v_1) \cdot n] \|_F,
\]

\[
F_{\text{PGEM}}(v_1) := \frac{1}{\sigma} a(\rho (f), \rho (v_1)),
\]

where, given \( \tau_F := \sqrt{\frac{e}{h_F}} \), \( \tau_F \) stands for

\[
\tau_F = \frac{e}{h_F} \Pi_F (b_F) = \frac{1}{\sqrt{e}} \frac{2b_F (1 - \cosh (x_F)) + h_F}{x_F \sinh (x_F)}.
\]

(40)

The new method (39) contains an edge-based CIP term which is balanced by a parameter \( \tau_F \) with no free constant (unlike the standard CIP term). The influence of edge-based stabilization on the numerical solutions for both (39) and a standard CIP method is explored numerically in Section 7.

\textbf{6. Recovering unusual stabilized methods}

Methods (9), (24), (38) and (39) possess terms which may have large gradients in the presence of boundary or internal layers. As such, special numerical quadratures [36] (or exact integration) are required to ensure their good performance. With the goal of avoiding the more involved numerical integration without losing accuracy, this section presents stabilized methods which are formally derived from the GEM (24).

Assume that the following approximation holds, \( K \in \mathcal{T}_h \),

\[
(\rho (w_1), \rho (v_1))_K \approx \sigma_{\text{max}} (w_1, v_1)_K \quad \text{for all} \quad v_1, w_1 \in S_1(K),
\]

(41)

where \( \sigma_{\text{max}} \in (0, 1) \) is the largest generalized eigenvalue \( \mu \) related to the generalized eigenvalue problem \( A w = \mu B w \). The entries of the symmetric positive definite matrices \( A \) and \( B \) are \( A_{ij} = (\rho (\psi_i), \rho (\psi_j))_K \) and \( B_{ij} = (\psi_i, \psi_j)_K \), respectively. We recall that \( \psi_i \), \( \psi_j \) are the standard (bi)linear basis for \( S_1(K) \). Furthermore, suppose

\[
e (\nabla \rho (w_1), \nabla \rho (v_1))_K \approx \varepsilon (\nabla w_1, \nabla v_1)_K \quad \text{for all} \quad v_1, w_1 \in S_1(K).
\]

(42)

Using approximations (41) and (42) with, alternatively, \( w_1 = u_1 \) and \( w_1 = f \), and replacing them in (24), the following stabilized method arises: \textit{Find} \( u_1 \in V_1(\Omega) \) such that

\[
B^F (u_1, v_1) = F^F (v_1) \quad \text{for all} \quad v_1 \in V_1(\Omega),
\]

(43)
with the piecewise function
\[ B(u_1, v_1) := a(u_1, v_1) - \sum_{K \in T_h} \tau_K (\sigma u_1, \sigma v_1)_K. \]
\[ F(v_1) := (f, v_1) - \sum_{K \in T_h} \tau_K (f, \sigma v_1)_K. \]

The parameter \[ \tau_K := \frac{1}{2} \left( 1 - \sigma \lambda_{\text{max}}^K \right) \] reads
\[ \tau_K = \begin{cases} \frac{1}{2} \left( 1 - \frac{6(4 \cosh(\|y\|) - 3 \sinh(\|y\|) - 3\|y\|^2 - 5)}{3\|y\| \sinh(\|y\|)} \right) & \text{for triangles,} \\ \frac{1}{2} \left( 1 - \frac{3 \sinh(\|y\|)^2 - \|y\|^2}{3\|y\| \sinh(\|y\|)} \right) & \text{for quadrangles.} \end{cases} \] (44)

A stabilized method starting from the PGEM (9) arises by assuming equivalent simplifications. This leads to the following stabilized method: Find \( u_1 \in V_1(\Omega) \) such that
\[ B_{\text{jump}}(u_1, v_1) := a(u_1, v_1) - \sum_{K \in T_h} \tau_K (\sigma u_1, \sigma v_1)_K, \]
\[ F_{\text{jump}}(v_1) := (f, v_1) - \sum_{K \in T_h} \tau_K (f, \sigma v_1)_K, \] (45)

where
\[ \tau_K = \begin{cases} \frac{1}{2} \left( 1 - \frac{6(4 \cosh(\|y\|) - 3 \sinh(\|y\|) - 3\|y\|^2 - 5)}{3\|y\| \sinh(\|y\|)} \right) & \text{for triangles,} \\ \frac{1}{2} \left( 1 - \frac{3 \sinh(\|y\|)^2 - \|y\|^2}{3\|y\| \sinh(\|y\|)} \right) & \text{for quadrangles.} \end{cases} \]

This version differs from (43) only in the definition of the elemental stabilization parameter. Figs. 4–6 illustrate the impact various values for \( \tau_K \) have on the enriching function \( \psi_i - \rho(\psi_i) \) (picted on a patch of four triangles) and the value of \( \tau_K \) given in (44).

**Remarks.**

- By following closely the equivalence result in Lemma 1 in [16] we can establish that (37) holds in the particular case \( w_1 = v_1 \), i.e. the norms induced by the bilinear forms are equivalent on the discrete space. The analysis of the error involved is not pursued here, though it will be addressed in a forthcoming work. For now, it is assumed that approximations (41) and (42) hold in the sense that the associated error is of the order of leading error estimates. This assumption is numerically verified in Section 7, wherein solutions from (24) (or (9)) and (43) (or (45)) are shown to stay "close".

- Using the approximation strategy in the jump-based version of the GEM (38), together with the assumption
\[ \frac{E}{h^p} (b_0 \| \nabla u_1 \cdot n \|, \epsilon [\nabla (\psi_i) \cdot n]_f) \]
\[ \sim \frac{E}{h^p} (b_0 \| \nabla u_1 \cdot n \|, \epsilon [\nabla (\psi_i) \cdot n]_f), \] (46)

the bilinear form \( B(\cdot, \cdot) \) in (43) includes the following:
\[ \sum_{K \in T_h} \tau_K ([\nabla u_1 \cdot n], \epsilon [\nabla (\psi_i) \cdot n]_f), \] (47)

where \( \tau_K \) is given in (40). The same extra term also appears when applying the approximation strategy to the jump-based version of PGEM (45). However, assumption (46) is no
longer necessary since (47) already appears explicitly in the method.

- Let \( V_k(\Omega) \) be the space of piecewise polynomial continuous functions of order \( k \) at most. A high-order version of stabilized method (43) may be written: Find \( u_k \) in \( V_k(\Omega) \) such that:

\[
B(u_k, v_k) = F(v_k) \quad \text{for all} \quad v_k \in V_k(\Omega),
\]

where

\[
\begin{align*}
B^e(u_k, v_k) & := a(u_k, v_k) - \sum_{K \in T_h} \tau_K (L u_k, L^* v_k)_K, \\
F^e(v_k) & := (f, v_k) - \sum_{K \in T_h} \tau_K (f, L^* v_k)_K
\end{align*}
\]

and \( L^* \) represents the adjoint operator of \( L \).
Method (43), together with (47), comprises of stabilization ingredients from the unusual and the edge-based stabilized methods. This is accomplished by the new design of elemental and edge stabilization parameters based on the enriching functions. Therefore, the method handles both the diffusive and the reactive limits continuously, with $\sigma_1$ vanishing for the former case and tending to one otherwise.

- All stabilized methods which have been presented are easily extended to 3D cases by redefining $\varepsilon_1$ with respect to the characteristic length of tetrahedrons (or hexahedrons). The methods also handle problems with variable physical coefficients by changing the value of $\varepsilon_1$ so that it is defined in terms of $\Pi^{l}(\varepsilon)$ and $\Pi^{r}(\sigma)$. This last claim is numerically verified in Section 7.

7. Numerical validations

With a series of three numerical tests, the GEM (24), its stabilized counterpart (43) (SGEM), and the stabilized version (45) of the PGEM (9) (denoted SPGEM) are shown to offer competitive, and often superior, results when compared with existing methods. All tests are performed on the unit square, which is partitioned into both structured (either triangles and rectangles) and unstructured triangular meshes. In particular, it is interesting to determine the effect of $\varepsilon_1 = O(h)$ as this has been shown to produce a localized resonance phenomenon [16,23] for the PGEM method (9) (see Fig. 3). Comparisons are made to four existing methods: 1) the PGEM (9), 2) the unusual stabilized finite element method (USFEM) of [6], 3) the CIP method of [10], and 4) the MsFEM (27). For the sake of completeness, the USFEM and the CIP methods are recalled.

The USFEM is: Find $u \in \mathcal{V}_h(\Omega)$ such that

![Figure 9. Profiles of solution from GEM and MsFEM at $x = 0.5$ Here $\varepsilon = 10^{-4}$.](image)

- Fig. 10. Solution from the new stabilized methods (45) and (43). Here $\varepsilon = 10^{-4}$.

- Fig. 11. Solutions from the USFEM (left) and the CIP (right) methods. Here $10^{-4}$.

![SPGEM](image)

![SGEM](image)

![USFEM](image)

![CIP](image)
Fig. 12. Deviation from the maximum principle. The structured (left) and the unstructured mesh cases (right).

Table 1
Relative local conservation errors. Here $\varepsilon = 10^{-6}$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>0.25</th>
<th>0.125</th>
<th>6.25 x 10^{-2}</th>
<th>3.125 x 10^{-2}</th>
<th>1.5625 x 10^{-2}</th>
<th>7.8125 x 10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>2.7 x 10^{-18}</td>
<td>4.4 x 10^{-17}</td>
<td>7 x 10^{-16}</td>
<td>10^{-14}</td>
<td>1.8 x 10^{-13}</td>
<td>2.7 x 10^{-12}</td>
</tr>
<tr>
<td>$M_2$</td>
<td>0.6667</td>
<td>0.6667</td>
<td>0.6667</td>
<td>0.6667</td>
<td>0.6667</td>
<td>0.6721</td>
</tr>
</tbody>
</table>

Fig. 13. Solutions from the PGEM and the GEM. Here $\varepsilon = 10^{-6}$.

Fig. 14. Solutions from the PGEM and the GEM. Here $\varepsilon = 10^{-4}$. 
\[ a(u_1, v_1) - \sum_{K \in T_h} \tau_K^{\text{USFEM}} (\sigma u_1, \sigma v_1)_K = (f, v_1)_\Omega - \sum_{K \in T_h} \tau_K^{\text{USFEM}} (\sigma v_1, f)_K, \quad \text{for all} \quad v \in V_1(\Omega) \quad (49) \]

where \( \tau_K^{\text{USFEM}} \) reads
\[ \tau_K^{\text{USFEM}} := \frac{h_k^2}{\sigma h_{	ext{max}}} \max \{1, \chi_k^{\text{USFEM}}\} + 6\varepsilon \]
and \( \chi_k^{\text{USFEM}} := \frac{\sigma}{\pi h_k^2} \)

As for the CIP method, it reads: Find \( u_1 \in V_1(\Omega) \) such that
\[ a(u_1, v_1) + \sum_{F \in \mathcal{E}} \tau^{\text{CIP}} (\nabla u_1 \cdot \mathbf{n}, \nabla v_1 \cdot \mathbf{n})_F = (f, v_1)_\Omega, \quad (50) \]
for all \( v \in V_1(\Omega) \), where \( \tau^{\text{CIP}} := \gamma h_\varepsilon^2 \), and \( \gamma \) is a positive constant set up following [10] equal to \( 10^{-2} \). Overall, we found out that the jump-based weak term (47) when included in the new methods lead to equivalent numerical results and, thus, they are not shown here.

---

**Fig. 15.** Solution from the new stabilized methods (45) and (43). Here \( \varepsilon = 10^{-4} \).

**Fig. 16.** Solutions from the USFEM and the CIP methods. Here \( \varepsilon = 10^{-4} \).

**Fig. 17.** Solutions from the PGEM and the present GEM. Here \( \varepsilon = 10^{-6} \).
7.1. The unit source problem

For this problem, the source $f = 1$, the reaction coefficient $\sigma = 1$ and homogenous Dirichlet boundary conditions are prescribed. Consider first a structured mesh made up of triangles. When $\varepsilon = 1$, all methods give similar results to the one from the Galerkin method.

In the highly reactive regime with $\varepsilon = 10^{-6}$, the PGEM and GEM solutions coincide as expected. Stabilized methods also work fine, although these still present small oscillations in two corners of $\Omega$.

An intermediate regime is considered by setting $\varepsilon = 10^{-4}$. As mentioned, the PGEM fails to completely suppress the oscillations plaguing the Galerkin’s solution. The GEM corrects this drawback, as depicted in Fig. 7. This is also achieved with the MsFEM as shown in Fig. 8, but the method over-predicts the ex-
act solution by a factor close to 100% (see Figs. 8 and 9). Additionally, the stabilized methods SPGEM and SGEM work similarly well with small imperfections near the boundary of the domain (Fig. 10).

It is interesting that the USFEM (49) produces results which are similar to those of the new stabilized methods, but the CIP method (50) is unable to correct the spurious modes (Fig. 11).

As a final study on the resonance error, we investigate how well the methods respect the maximum principle, which imposes the bound $u(x) \leq 1$ for all $x \in \Omega$. To this end, we measure the maximum norm $\max_{x \in \Omega} |u(x)|$ with respect to small values of $\epsilon$ for different methods on a criss-cross and on an unstructured mesh with around 400 triangles. We can see a great improvement in adopting the GEM when compared to the other methods in Fig. 12.

In Table 1, we study the local conservation property of the GEM (24), considering alternatively $u_1$ or $u_1 + u_1$. Define the quantities

$$M_2 := \max_{x \in \Omega} \frac{\int_K \mathcal{L}(u_1 + u_1) - fx}{|K|} \quad \text{and} \quad M_1 := \max_{x \in \Omega} \frac{\int_K \mathcal{L}u_1 - fx}{|K|}$$

As expected, we observe significant losses when considering only the linear part of the solution. Nevertheless, we recover the local

---

**Fig. 21.** Error between $u$ and $u_1$ in the relative $H^1(\Omega)$ semi-norm obtained from different methods. Here $\epsilon = 10^{-6}$.

**Fig. 22.** Errors between $u$ and $u_1$, and $u_0$ from the GEM (24) with respect to $h$ in the relative $H^1(\Omega)$ semi-norm (left). Error between $u$ and $u_0$ from the GEM (24) and PGEM (9) with respect to $\epsilon$ in the $H^1(\Omega)$ semi-norm (right).

**Fig. 23.** Solution from the Galerkin method. Here $\epsilon = 10^{-6}$. 
conservation property updating the function $u_i$ by the multiscale enrichment $u_\omega$.

Next, an unstructured mesh is employed to examine how sensitive the methods are to the mesh. The conclusions reached in the case of a structured mesh are still valid. It is worth mentioning that the GEM produces the best results for all regimes. See Fig. 13 for $\varepsilon = 10^{-6}$ and Figs. 14–16 for the case $\varepsilon = 10^{-4}$.

Fig. 24. Solutions from the GEM, its associated stabilized method, the USFEM and the CIP method. Here $\varepsilon = 10^{-6}$.

Fig. 25. Profile of solutions at $x = 0.5$. Here $\varepsilon = 10^{-6}$. 
Next, the methods are validated on a structured quadrilateral mesh. For the highly reactive case $\varepsilon = 10^{-6}/C_0$, both the GEM and PGEM correct oscillations (see Fig. 17). The SGEM and the SPGEM also give acceptable results. Comparing the USFEM solution to that of the stabilized GEM, similar behavior is present. On the other hand, the solution from the CIP method still suffers from spurious oscillations (Fig. 18).

As before, the case $\varepsilon = 10^{-4}$ is also considered. The conclusions reached concerning the GEM and the PGEM are once again valid.

Fig. 26. Zoom of profiles along two portions: $y \in [0, 0.25]$ (left) and $y \in [0.35, 0.65]$ (right). Here $\varepsilon = 10^{-6}$.

Fig. 27. Solutions from the Galerkin method, the GEM and the PGEM. Here $\varepsilon = 10^{-4}$.

Galerkin

PGEM

GEM
i.e., the GEM is able to correct the oscillations present in the solution of the PGEM (Fig. 19). On the other hand, the stabilized method derived from the GEM is more precise than that derived from the PGEM, with only small oscillations at the corners (Fig. 20).

In order to verify the assertion that the GEM outperforms the other methods tested in a quantifiable manner, we have measured the error between numerical solutions and an "exact" solution calculated using the standard Galerkin method on a very fine structured quadrilateral mesh (1024 × 1024 elements).

First, the measurement is carried out in the relative $H^1(\Omega)$ semi-norm $\|u - u_h\|_{H^1(\Omega)}$, where $u_1$ stands for the solutions from the different methods and $u$ is the "exact" solution. We see in Fig. 21 that the GEM (24) shows an improved performance when compared to standard Galerkin method (5) and the CIP method (50) (here $\varepsilon = 10^{-6}$). Also, stabilized method SGEM (43) originated from GEM (24) preserves the GEM’s accuracy despite the approximations involved in its construction.

Next, we take advantage of the enriching space framework to update the solution $u_1$ from GEM (24) and PGEM (9) with their enriched counterpart $u_e$ given in (15) and (16). As a result, the error is drastically decreased in the relative $H^1(\Omega)$ semi-norm by using the updated solution $u_h := u_1 + u_e$ instead of $u_1$, as shown in Fig. 22.

Also, we look at the error $|u - u_h|_{H^1(\Omega)}$ in order to compare GEM (24) to PGEM (9) with respect to $\varepsilon$. We point out in Fig. 22 that the GEM outperforms the PGEM when $\varepsilon \sim 10^{-4}$ (here $h = \frac{1}{16}$).

7.2. The discontinuous source problem

The intent of this test is to validate the methods when a discontinuous source $f$ induces internal layers. Let $f = 1$ for $y \in [0,0.5)$ and $f = 0$ for $y \in [0.5,1]$, and $\sigma = 1$. Consider the case of a structured triangulation of the domain. In the intermediate case $\varepsilon = 10^{-4}$, results are equivalent to those obtained from the previous numerical tests, i.e., the GEM overcomes the resonance difficulty of the PGEM, while the results of stabilized methods are equivalent.

For the case $\varepsilon = 10^{-6}$, in Fig. 23 we depict the oscillatory Galerkin solution. Next, in Fig. 24 we show that the stabilized method originating from the GEM (43) induces more numerical diffusion across the internal layer than the GEM itself (this was also present for $\varepsilon = 10^{-4}$). A similar result holds for the PGEM and its stabilized method counterpart (45). Also, Fig. 24 shows that the USFEM produces similar results as the new stabilized method while the CIP method is again unable to correct oscillations on external layers. Indeed, Figs. 25 and 26 summarize the results by depicting the
cross sections of solutions from different methods, highlighting the accuracy of the GEM. The MsFEM presents, once again, great over-approximation when compared to the GEM (see Fig. 25). Next, we consider the same problem with \( \varepsilon = 10^{-6} \), but on an unstructured mesh. Once again, the numerical results from the GEM and the PGEM coincide. We also explore the case \( \varepsilon = 10^{-4} \). The results depicted in Fig. 27 show that the GEM is oscillation-free even for this challenging numerical test, whereas the more classical methods fail to introduce the correct numerical diffusion. In particular, we observe once again that the GEM outperforms the PGEM.

7.3. The variable coefficient case

We now suppose the reaction coefficient \( \sigma = \sigma(x) \) varies as \( 1 - x - y \) in order to study how the methods behave in the presence of a non-constant reaction. Regarding the enriching basis functions, accuracy is preserved when they are defined as the solution of a non-constant reaction. Regarding the enriching basis functions, accuracy is preserved when they are defined as the solution of a non-constant reaction. The MsFEM presents, once again, great overapproximation when compared to the GEM (see Fig. 25).

8. Conclusion

Enriching both trial and test spaces with multiscale functions, a new Galerkin enriched finite element method (GEM) has been produced which minimizes spurious oscillations for all asymptotic regimes. In addition, new stabilized finite elements originating with the GEM and the PGEM are derived. These new stabilized methods are constant-free and avoid more involved numerical integrations while maintaining accuracy. The numerical experiments have shown, in particular, that edge-based stabilization is unnecessary (or deleterious in the case of CIP methods) and the MsFEM is inadequate to correctly approach reactive boundary layers. All together, results show the new methods are competitive and outperform more classical stabilized and enriched methods for the reaction-diffusion model.

References